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Energy Gap, Clustering, and the Goldstone Theorem in Statistical Mechanics

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We prove a Goldstone-type theorem for a wide class of lattice and continuum quantum systems, both for the ground state and at nonzero temperature. For the ground state ($T = 0$) spontaneous breakdown of a continuous symmetry implies no energy gap. For nonzero temperature, spontaneous symmetry breakdown implies slow clustering (no L^1 clustering). The methods apply also to nonzero-temperature classical systems.

KEY WORDS: Energy gap; clustering; Goldstone theorem.

1. INTRODUCTION

Given a physical system with short-range forces and a continuous symmetry, if the ground state is not invariant under the symmetry the Goldstone theorem states that the system possesses excitations of arbitrarily low energy.[1,2] In the case of the ground state (vacuum) of local quantum field theory, the existence of an energy gap is equivalent to exponential clustering.[3] In this framework the Goldstone theorem was proved in Refs. 4 and 5. For general ground states of nonrelativistic systems, the two properties (energy gap and clustering) are, however, independent and, in particular, the assumption that the ground state is the unique vector invariant under time translations does not necessarily follow from the assumption of spacelike clustering, as remarked in Ref. 6. This point was not taken into account in the assumptions of Refs. 7 and 8. Another related aspect, of greater relevance to our discussion, is the fact that the rate of clustering is

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not expected to be related to symmetry breakdown and absence of an energy gap, since for example the ground state of the Heisenberg ferromagnet[6] has a broken symmetry and no energy gap, but is exponentially clustering (for the ground state is a product state of spins pointing in a fixed direction). On the other hand, for $T > 0$ no energy gap is expected to occur, at least under general timelike clustering assumptions (Ref. 12, Proposition 3); these assumptions may be verified for the free Bose gas.[13]

At nonzero temperature it is the cluster properties that are important in connection with symmetry breakdown. At nonzero temperature we may then formulate the Goldstone theorem as follows. Given a system with short-range forces and a continuous symmetry, if the equilibrium state is not invariant under the symmetry, then the system does not possess exponential clustering.

It is our purpose to explore the validity of the Goldstone theorem for a wide class of spin systems and many-body systems, both for the ground state and at nonzero temperature. The main tool we will use at nonzero temperature is the Bogoliubov inequality, which is valid for both classical and quantum systems (see, for example, Ref. 9). We shall, however, present the discussion in the framework of quantum statistical mechanics. At zero temperature our method is related to that of Ref. 7, where a version of the theorem was proved, valid for one space dimension. A different proof, valid for the ferromagnetic Heisenberg Hamiltonian of finite range, and in greater analogy to the quantum field theory proofs of Refs. 4 and 5, was given in Ref. 6, and generalized in Ref. 8 to quantum spin systems of finite range. Our methods apply also to nonzero temperature classical systems, for which related results were obtained in Refs. 14 and 15.

Our results apply to states which are invariant with respect to spatial translations by some discrete set which is sufficiently dense. (For lattice systems this could be a sublattice, and for continuum systems, a lattice imbedded in the continuum.) More precisely, we require the following condition.

Condition $\mathfrak{L}$. There is a constant l such that for all sufficiently large cubes Λ,

$$|\Lambda_{\mathfrak{L}}|/|\Lambda| \geqslant l$$

where $|\Lambda|$ is the volume of Λ and $|\Lambda_{\mathfrak{L}}|$ is the number of points in $\Lambda_{\mathfrak{L}} = \Lambda \cap \mathfrak{L}$.

We will prove that for interactions which are not too long range (see Sections 3 and 4 for examples), for the ground state ($T = 0$) spontaneous breakdown of a continuous symmetry implies no energy gap. For nonzero temperature ($T > 0$) spontaneous symmetry breakdown implies no exponential clustering (in fact no L^1 clustering).

For a continuous system our results cover the case of the breakdown of translational invariance. However, at $T = 0$ and nonzero densities there is never an energy gap due to the breakdown of Galilean invariance as remarked in Ref. 11.

Finally, we should like to stress that, although we present an informal treatment of the continuum case, our results for quantum spin systems are complete and rigorous.

2. GENERAL FRAMEWORK

The state of the system is described by the vector Ω in some Hilbert space. There is a symmetric Hamiltonian operator H and $H\Omega = 0$. To each cube $\Lambda \subset \mathbb{R}^d$ there is a set of observables $\mathfrak{A}_\Lambda$ such that $[A, B] = 0$ if $A \in \mathfrak{A}_{\Lambda_1}$, $B \in \mathfrak{A}_{\Lambda_2}$ and Λ_1, Λ_2 are disjoint. The set of all observables is $\mathfrak{A} = \bigcup_\Lambda \mathfrak{A}_\Lambda$. We define $\hat{A} = A - (\Omega, A\Omega)$. We suppose $A\Omega$ is in the domain of H for all $A \in \mathfrak{A}$.

Let τ_x denote spatial translation by x. (In the continuum case $x \in \mathbb{R}^d$ in the lattice case $x \in \mathbb{Z}^d$.) The state Ω is invariant under translations in the discrete set $\mathfrak{L}$, satisfying condition $\mathfrak{L}$ of the introduction. Thus

$$(\Omega, \tau_i A\Omega) = (\Omega, A\Omega) \qquad \forall A \in \mathfrak{A}, \qquad \forall_i \in \mathfrak{L}$$

There is a one-parameter group of symmetry transformations σ_s of $\mathfrak{A}$ commuting with the Hamiltonian and with all spatial translations:

$$\sigma_s H A\Omega = H\sigma_s A\Omega$$
$$\sigma_s \tau_x = \tau_x \sigma_s \qquad \forall x \in \mathbb{R}^d \quad (\forall x \in \mathbb{Z}^d) \tag{2.1}$$

We suppose the symmetry σ_s is generated by a current

$$J_x = \tau_x J_0$$

where $J_0 \in \mathfrak{A}_0$ (lattice case); $J_0 \in \mathfrak{A}_\Delta$ (continuum case); and Δ is a cube of side δ. (In the continuum case we may suppose J_x is smooth in x by first averaging the current over the small cube Δ.)

Thus, if $A \in \mathfrak{A}_\Lambda$

$$\frac{d}{ds}\bigg|_{s=0} (\Omega, \sigma_s A\Omega) = i(\Omega, [J_\Lambda, A]\Omega)$$

where

$$J_\Lambda = \sum_{i \in \Lambda} J_i \qquad \text{(lattice case)}$$

$$J_\Lambda = \int_{\Lambda_\delta} d^d x\, J_x \qquad \text{(continuum case)}$$

and Λ_δ is a set of points within distance δ from Λ. By the group property, the invariance of Ω under the symmetry σ_s follows from

$$(\Omega, [J_\Lambda, A]\Omega) = 0$$

for all $A \in \mathfrak{A}_\Lambda$, and all cubes Λ. The equilibrium property of Ω is given as follows:

(a) $T = 0$: Ω is a ground state; i.e., $(\psi, H\psi) \geqslant 0$ for all ψ in the domain of H.

(b) $T > 0$: Ω satisfies the Bogoliubov inequality; i.e., for all $A, B \in \mathfrak{A}$,

$$|(\Omega, [B, A]\Omega)|^2 \leqslant \beta \qquad \left(\Omega, \tfrac{1}{2}(A^\dagger A + AA^\dagger)\Omega\right) \frac{1}{i} \left(\Omega, [B^\dagger, \dot{B}]\Omega\right)$$

where $\dot{B} = i[H, B]$. Note that $(1/i)(\Omega, [B^\dagger, \dot{B}]\Omega)$ may be written in the form $(B\Omega, HB\Omega) + (B^\dagger\Omega, HB^\dagger\Omega)$.

The basic hypothesis about the state Ω which leads to the absence of symmetry breaking is as follows:

(a) $T = 0$: there is an energy gap $\epsilon > 0$, i.e., $(\psi, H\psi) \geqslant \epsilon$ for all Ψ in the domain of H, orthogonal to Ω, $\|\psi\| = 1$.

(b) $T > 0$: there is L^1 clustering; i.e., for each observable A,

$$\sum_{i \in \mathfrak{L}} |(\Omega, A^\dagger \tau_i A\Omega) - (\Omega, A^\dagger\Omega)(\Omega, \tau_i A\Omega)| < \infty$$

The basic strategy is as follows. For each self-adjoint observable $A \in \mathfrak{A}_{\Lambda_0}$ define $\gamma = d/ds|_{s=0}(\Omega, \sigma_s A\Omega)$. We must show $\gamma = 0$. Now by the translation invariance of Ω with respect to $\mathfrak{L}$ we have for each cube Λ (and $\sigma_s \tau_i = \tau_i \sigma_s$)

$$\gamma = \frac{1}{|\Lambda_{\mathfrak{L}}|} \frac{d}{ds}\bigg|_{s=0} \left(\Omega, \sigma_s \sum_{i \in \Lambda_{\mathfrak{L}}} \tau_i A\Omega\right)$$

where $\Lambda_{\mathfrak{L}} = \Lambda \cap \mathfrak{L}$ and $|\Lambda_{\mathfrak{L}}|$ is the number of points in $\mathfrak{L}$.

Thus

$$\gamma = \frac{1}{|\Lambda_{\mathfrak{L}}|} i \left(\Omega, \left[\hat{J}_{\tilde{\Lambda}}, \sum_{j \in \Lambda_{\mathfrak{L}}} \tau_j \hat{A}\right]\Omega\right)$$

where

$$\tilde{\Lambda} = \bigcup_{i \in \Lambda} \tau_i \Lambda_0$$

We estimate γ as follows:

(a) $T = 0$:

$$|\gamma|^2 \leqslant \frac{4}{|\Lambda_{\mathfrak{L}}|^2} (\hat{J}_{\tilde{\Lambda}}\Omega, \hat{J}_{\tilde{\Lambda}}\Omega) \left(\sum_{i \in \Lambda_{\mathfrak{L}}} \tau_i \hat{A}\Omega, \sum_{j \in \Lambda_{\mathfrak{L}}} \tau_j \hat{A}\Omega\right)$$

Now for any observable B, $(\Omega, \hat{B}\Omega) = 0$. Thus the assumption of an energy gap $\epsilon > 0$ implies

$$(\hat{B}\Omega, \hat{B}\Omega) \leqslant \frac{1}{\epsilon}(\hat{B}\Omega, H\hat{B}\Omega)$$
$$\leqslant \frac{1}{\epsilon}\left[(\hat{B}\Omega, H\hat{B}\Omega) + (\hat{B}^{\dagger}\Omega, H\hat{B}^{\dagger}\Omega)\right]$$
$$= \frac{1}{\epsilon}\left(\Omega, \left[B^{\dagger}, \left[H, B\right]\right]\Omega\right) = \frac{1}{i\epsilon}\left(\Omega, \left[B^{\dagger}, \dot{B}\right]\Omega\right)$$

Thus

$$|\gamma|^2 \leqslant \frac{4}{\epsilon}\left[\frac{1}{|\Lambda_{\mathfrak{L}}|}\frac{1}{i}\left(\Omega, \left[J_{\tilde{\Lambda}}, \dot{J}_{\tilde{\Lambda}}\right]\Omega\right)\right] \cdot \frac{1}{|\Lambda_{\mathfrak{L}}|}\left\|\sum_{j\in\Lambda_{\mathfrak{L}}} \tau_j \hat{A}\Omega\right\|^2 \qquad (*)$$

(b) $T > 0$: using the Bogoliubov inequality,

$$|\gamma|^2 \leqslant \beta\left[\frac{1}{|\Lambda_{\mathfrak{L}}|}\frac{1}{i}\left(\Omega, \left[J_{\tilde{\Lambda}}, \dot{J}_{\tilde{\Lambda}}\right]\Omega\right)\right]\frac{1}{|\Lambda_{\mathfrak{L}}|}\left\|\sum_{j\in\Lambda_{\mathfrak{L}}} \tau_j \hat{A}\Omega\right\|^2 \qquad (**)$$

Notice the similarity of inequalities (*) and (**).

In one case the coefficient involves β, the inverse temperature. In the other case the coefficient involves $1/\epsilon$, the inverse gap.

To prove the absence of symmetry breakdown we will show in both cases ($T = 0$ and $T > 0$)

$$\text{(I)} \qquad \frac{1}{|\Lambda_{\mathfrak{L}}|}\left(\Omega, \left[J_{\tilde{\Lambda}}, \dot{J}_{\tilde{\Lambda}}\right]\Omega\right) \to 0 \qquad \text{as} \quad \Lambda \nearrow \mathbb{R}^d$$

and

$$\text{(II)} \qquad \frac{1}{|\Lambda_{\mathfrak{L}}|}\left\|\sum_{j\in\Lambda_{\mathfrak{L}}} \tau_j \hat{A}\Omega\right\|^2 \leqslant C \qquad \text{uniformly in } \Lambda$$

(I) follows essentially from properties of the Hamiltonian. (II) follows from L^1 clustering ($T > 0$) or from properties of the Hamiltonian and the energy gap ($T = 0$). Indeed from L^1 clustering, and invariance of Ω under τ_j, $j \in \mathfrak{L}$,

$$\frac{1}{|\Lambda_{\mathfrak{L}}|}\left\|\sum_{j\in\Lambda_{\mathfrak{L}}} \tau_j \hat{A}\Omega\right\|^2 \leqslant \sum_{j\in\mathfrak{L}} |(\Omega, A\tau_j A\Omega) - (\Omega, A\Omega)(\Omega, A\Omega)| < \infty$$

In the case $T = 0$, from the energy gap ϵ, we have

$$\frac{1}{|\Lambda_{\mathfrak{L}}|}\left\|\sum_{\tau\in\Lambda_{\mathfrak{L}}} \tau_j \hat{A}\Omega\right\|^2 \leqslant \frac{1}{i\epsilon}\frac{1}{|\Lambda_{\mathfrak{L}}|}\left(\Omega, \left[\sum_{j\in\Lambda_{\mathfrak{L}}} \tau_j A, \sum_{k\in\Lambda_{\mathfrak{L}}} \tau_k \dot{A}\right]\Omega\right)$$
$$\leqslant \frac{1}{i\epsilon}\sum_{k\in\mathfrak{L}} \left|\left(\Omega, \left[A, \tau_k \dot{A}\right]\Omega\right)\right|$$

The finiteness of the sum over $\mathfrak{L}$ will follow from properties of the Hamiltonian.

The system is said to have *property* G_T if $\sup_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}|(\Omega,[J_j,\dot{J}_k]\Omega)|\to 0|$ $< \infty$

$$\sup_{j\in\mathbb{Z}^d}\sum_{\substack{k\in\mathbb{Z}^d\\|k-j|\geq D}}|(\Omega,[J_j,\dot{J}_k]\Omega)|\to 0 \qquad \text{as } D\nearrow\infty$$

or

$$\sup_{x\in\mathbb{R}^d}\int d_y^d|(\Omega,[J_x,\dot{J}_y]\Omega)|<\infty \qquad \text{and}$$

$$\sup_{x\in\mathbb{R}^d}\int_{|y-x|\geq D} d^dy\,|(\Omega,[J_x,\dot{J}_y]\Omega)|\to 0 \qquad \text{as } D\uparrow\infty$$

The system is said to have *property* G_0 if property G_T holds and for each self-adjoint observable A,

$$\sum_{k\in\mathfrak{L}}|(\Omega,[A,\tau_k\dot{A}]\Omega)|<\infty$$

We may now state Goldstone's theorem in the following form:

Theorem 1. (a) $T=0$: If the system possesses an energy gap and property G_0 then there is no spontaneous symmetry breakdown.

(b) $T>0$: If the system possesses L^1 clustering and property G_T then there is no spontaneous symmetry breakdown.

Proof. We must show that (I) follows from

$$\sup_{x\in\mathbb{R}^d}\int_{|y-x|\geq D} d^dy\,|(\Omega,[J_x,\dot{J}_y]\Omega)|\underset{D\nearrow\infty}{\longrightarrow}0$$

(The lattice case is analogous.) Write $\mathcal{J}(x,y)=(\Omega,[J_x,\dot{J}_y]\Omega)$. Then, by property G_T,

$$\int d^dy\,|\mathcal{J}(x,y)|<\infty \quad \text{and} \quad \sup_{x\in\mathbb{R}^d}\int_{|y-x|\geq D} d^dy\,|\mathcal{J}(x,y)|\underset{D\nearrow\infty}{\longrightarrow}0$$

Now

$$\int d^dx\,\mathcal{J}(x,y)=\frac{d}{ds}\Big|_{s=0}(\Omega,\sigma_sHJ_y\Omega)=\frac{d}{ds}\Big|_{s=0}(\Omega,H\sigma_sJ_y\Omega)=0 \quad (***)$$

Also by the Jacobi identity $\mathcal{J}(x,y)=\mathcal{J}(y,x)$. Then since $|\Lambda_{\mathfrak{L}}|\geq l|\Lambda|$ $\geq(l/|\Lambda_0|)|\tilde{\Lambda}|$ we estimate

$$\frac{1}{|\tilde{\Lambda}|}(\Omega,[J_{\tilde{\Lambda}},\dot{J}_{\tilde{\Lambda}}]\Omega)=\frac{1}{|\tilde{\Lambda}|}\int d^dx\,d^dy\,\chi_{\tilde{\Lambda}_\delta}(x)\chi_{\tilde{\Lambda}_\delta}(y)\mathcal{J}(x,y)$$

$$=-\frac{1}{2|\tilde{\Lambda}|}\int d^dx\,d^dy|\chi_{\tilde{\Lambda}_\delta}(x)-\chi_{\tilde{\Lambda}_\delta}(y)|^2\mathcal{J}(x,y)$$

where χ_Λ is the characteristic function of Λ, using (***). Thus

$$\left|\frac{1}{|\tilde{\Lambda}|}\left(\Omega,\left[J_{\tilde{\Lambda}},\dot{J}_{\tilde{\Lambda}}\right]\Omega\right)\right| \leqslant \frac{1}{|\tilde{\Lambda}|}\int_{\tilde{\Lambda}_\delta} d^dx \int_{\tilde{\Lambda}_\delta^c} d^dy\, |\mathfrak{H}(x,y)| \equiv (2.2)$$

We write

$$\int_{\tilde{\Lambda}_\delta} d^dx = \int_{\Lambda_1} d^dx + \int_{\Lambda_2} d^dx$$

where Λ_2 is the set of points in $\tilde{\Lambda}_\delta$ within distance D of the boundary and $\Lambda_1 = \tilde{\Lambda}_\delta - \Lambda_2$.

Then

$$(2.2) \leqslant \sup_x \int_{|x-y|\geqslant D} d^dy\, |\mathfrak{H}(x,y)| + \frac{|\Lambda_2|}{|\tilde{\Lambda}|} \sup_x \int d^dy\, |\mathfrak{H}(x,y)|$$

The first term goes to zero by property G_T and the second term goes to zero since

$$\frac{|\Lambda_2|}{|\tilde{\Lambda}|} \underset{\tilde{\Lambda}\nearrow\mathbb{R}^d}{\rightarrow} 0$$

The theorem gains content by analyzing when property G_T or G_0 holds. This will be done in the following sections.

3. QUANTUM SPIN SYSTEMS

The interaction Φ is determined by specifying for each finite $X \in \mathbb{Z}^d$ the "connected" X-body interaction $\Phi(X) \in \mathfrak{A}_X$. The Hamiltonian H is then defined by

$$HA\Omega = \sum_{x\cap\Lambda_0\neq\emptyset} \left[\Phi(X), A\right]\Omega$$

for $A \in \mathfrak{A}_{\Lambda_0}$. Let $D(X)$ denote the diameter of X:

$$D(X) \equiv \sup_{k,j\in X} |i-j|$$

Theorem 2. The system satisfies G_T and G_0 if

$$\sup_i \sum_{x\ni i} |X|\, \|\Phi(x)\| < \infty$$

and

$$\sup_i \sum_{\substack{X\ni i\\ D(X)\geqslant d}} |X|\, \|\Phi(X)\| \rightarrow 0 \qquad \text{as} \quad d\nearrow\infty$$

Note that if the interaction is translation invariant $[\Phi(X+i) = \tau_i\Phi(X)]$

then the above follows from

$$\sum_{x \ni 0} |X| \, \|\Phi(X)\| < \infty$$

If furthermore the interaction is at most N-body $[\Phi(X) = 0$ if $|X| > N]$ then the above is equivalent to $\sum_{x \ni 0} \|\Phi(X)\| < \infty$. In particular for the Heisenberg Hamiltonian $\sum J(i-j)\sigma_i\sigma_j$ we require $\sum_{i \in \mathbb{Z}^d} |J(i)| < \infty$.

Proof. (a) We must show

$$\sup_{j \in \mathbb{Z}^d} \sum_{\substack{k \in \mathbb{Z}^d \\ |k-j| \geq D}} \left|\left(\Omega, \left[J_j, \dot{J}_k\right]\Omega\right)\right| \underset{D \nearrow \infty}{\longrightarrow} 0$$

Now

$$\sum_{\substack{k \\ |k-j| \geq D}} \left\|\left[J_j, \dot{J}_k\right]\right\| \leq \sum_{\substack{k \\ |k-j| \geq D}} \sum_{x \ni k, j} \left\|\left[J_j, \left[\Phi(X), J_k\right]\right]\right\|$$

$$\leq 4\|J_0\|^2 \sum_{\substack{k \\ |k-j| \geq D}} \sum_{x \ni k, j} \|\Phi(X)\| \leq 4\|J_0\|^2 \sum_{\substack{x \ni j \\ D(X) \geq D}} |X| \, \|\Phi(X)\|$$

which goes to zero as $D \nearrow \infty$ by the hypothesis of the theorem.

(b) We will show $\sum_{j \in \mathbb{Z}^d} |(\Omega, [A, \tau_j \dot{A}]\Omega)| < \infty$ for each observable A. The proof is similar to (a). Let $A \in \mathfrak{A}_{\Lambda_0}$.

$$\sum_{j \in \mathbb{Z}^d} \left\|\left[A, \tau_j \dot{A}\right]\right\| \leq 4\|A\|^2 \sum_{j \in \mathbb{Z}^d} \sum_{\substack{X \cap \tau_j \Lambda_0 \neq \emptyset \\ X \cap \Lambda_0 \neq \emptyset}} \|\phi(X)\|$$

$$+ 4\|A\|^2 \sum_{\substack{j \\ \Lambda_0 \cap \tau_j \Lambda_0 \neq \emptyset}} \sum_{X \cap \tau_j \Lambda_0 \neq \emptyset} \|\phi(X)\|$$

$$\leq 4\|A\|^2 |\Lambda_0| \sup_i \sum_{X \ni i} |X| \, \|\phi(X)\|$$

$$+ 4\|A\|^2 |\Lambda_0|^2 |\Lambda_0| \sup_i \sum_{X \ni i} \|\phi(X)\|$$

$$\leq 8\|A\|^2 |\Lambda_0|^3 \sup_i \sum_{X \ni i} |X| \, \|\Phi(X)\| \qquad \blacksquare$$

4. CONTINUUM SYSTEMS

In this section we will proceed in an informal way, emphasizing the procedure and type of estimates. In the continuum case unbounded operators will arise and, having constructed a particular equilibrium state, it would be necessary to verify that the correlation functions are indeed well defined.

The Goldstone theorem is applicable to local quantum fields at zero and nonzero temperature. Indeed, properties G_T and G_0 are immediate consequences of the finite propagation speed.

Nonrelativistic many-body systems may also be treated. We consider first the breakdown of an internal symmetry in a translation-invariant system at nonzero temperature. The Bose and Fermi cases are treated in the same way. A particle with M internal degrees of freedom is described by the field operators

$$\Psi_j(x), \qquad j = 1, 2, \ldots, M$$

which satisfy

$$\left[\Psi_j(x), \Psi_k^\dagger(y)\right]_\pm = \delta_{jk}\delta(x-y)$$

Let $\sigma^1, \ldots, \sigma^g$ be the self-adjoint $M \times M$ matrices of a representation of the Lie algebra of a compact semisimple Lie group $\mathcal{G}$ with (totally antisymmetric[10]) structure constants:

$$\left[\sigma^\alpha, \sigma^\beta\right] = i\Gamma_\gamma^{\alpha\beta}\sigma^\gamma$$

Define

$$\rho^\alpha(x) = \sum_{jk} \psi_j^\dagger(x)\sigma_{jk}^\alpha\psi_k(x)$$

We also write this as $\psi^\dagger(x)\sigma^\alpha\Psi(x)$. The operators $\rho^\alpha(x)$ are the local generators of the Lie group transformations on the fields $\Psi_k(x)$.

The Hamiltonian has the form $H = H_0 + V - \mu N$, where

$$H_0 = \frac{1}{2m}\sum_j \int d^n x\ \underline{\nabla}\psi_j^\dagger(x)\cdot\underline{\nabla}\Psi_j(x)$$

$$V = \sum_\alpha \int d^n x\, d^n y : \rho^\alpha(x)\rho^\alpha(y) : V(x-y)$$

$$N = \sum_j \int d^n x\ \Psi_j^\dagger(x)\Psi_j(x)$$

If the internal symmetry $\mathcal{G}$ is spontaneously broken, then the absence of L^1-clustering would follow from

$$\int dy \left|\left(\Omega, \left[J_0, \dot{J}_y\right]\Omega\right)\right| < \infty$$

where

$$J_y = \int d^n x\, h(y-x)\rho^\alpha(x)$$

and h is a smooth function of compact support and

$$\int d^n x\, h(x) = 1$$

With

$$\mathbf{J}^\alpha(x) = \frac{1}{2i}\left(\Psi^\dagger\sigma^\alpha\underline{\nabla}\Psi - \underline{\nabla}\Psi^\dagger\sigma^\alpha\Psi\right)$$

we have the following algebraic relations which hold in both the Bose and Fermi case:

$$[\rho^{\alpha}(x),\rho^{\beta}(y)] = i\delta(x-y)\Gamma_{\gamma}^{\alpha\beta}\rho^{\alpha}(y)$$
$$i[H_0,\rho^{\alpha}(x)] = -\underline{\nabla}\cdot\mathbf{J}^{\alpha}(x)$$
$$[N,\rho^{\alpha}(x)] = 0$$
$$[\rho^{\alpha}(x),\mathbf{J}^{\beta}(y)] = \frac{1}{i}\underline{\nabla}_x\left[\Psi^{\dagger}(x)\frac{[\sigma^{\alpha},\sigma^{\beta}]}{2} + \Psi(x)\delta(x-y)\right] + i\Gamma_{\gamma}^{\alpha\beta}\mathbf{J}^{\gamma}(y)\delta(x-y)$$
$$[\rho^{\alpha}(y),[\rho^{\alpha}(x),V]] = \delta(y-x)\sum_{\beta\gamma\delta}\Gamma_{\gamma}^{\alpha\beta}\Gamma_{\delta}^{\alpha\beta} \times\int dz\, V(x-z) : [\rho^{\delta}(x),\rho^{\gamma}(z)]+: - \sum_{\beta\gamma\delta}\Gamma_{\gamma}^{\alpha\beta}\Gamma_{\delta}^{\alpha\beta}V(x-y) : [\rho^{\gamma}(x),\rho^{\delta}(y)]+:$$

where we have used

$$\rho^{\alpha}(x)\rho^{\beta}(y) := \rho^{\alpha}(x)\rho^{\beta}(y) - \delta(x-y)\Psi^{\dagger}(y)\sigma^{\alpha}\sigma^{\beta}\Psi(y)$$

To show

$$\int dy\,|(\Omega,[J_0,\dot{J}_y]\Omega)| < \infty$$

we need only consider the nonlocal term

$$\sum_{\beta\gamma\delta}\Gamma_{\gamma}^{\alpha\beta}\Gamma_{\delta}^{\alpha\beta}V(x-y)(\Omega,\ :\rho^{\gamma}(x)\rho^{\delta}(y):\Omega)$$

So if

$$\int dy\,|V(x-y)|\,|(\Omega,\ :\rho^{\gamma}(x)\rho^{\delta}(y):\Omega)| < \infty$$

the result follows.

We note that there is a qualitative difference between Abelian and non-Abelian groups since in the Abelian case $\Gamma_{\gamma}^{\alpha\beta} = 0$ and only local terms occur. {A similar analysis applies to the $T = 0$ case, although here the distinction between Abelian and non-Abelian groups does not arise because the term $(\Omega,[A,\tau_j\dot{A}]\Omega)$ will in general have a term with $V(x-y)$ even in the Abelian case.}

We consider now the breakdown of translation invariance. We will for simplicity not consider internal degrees of freedom, so that now

$$H = H_0 + V - \mu N$$

where

$$H_0 = \frac{1}{2m} \int d^n x \, \underline{\nabla} \Psi^\dagger(x) \cdot \underline{\nabla} \Psi(x)$$

$$\rho(x) = \Psi^\dagger(x) \Psi(x)$$

$$N = \int d^n x \, \rho(x)$$

$$V = \int d^n x \, d^n y \, V(x-y) : \rho(x) \rho(y):$$

The local generator of the translation group is

$$\mathbf{j}(x) = \frac{1}{2i} \left[\Psi^\dagger(x) \nabla \Psi(x) - \nabla \Psi^\dagger(x) \Psi(x) \right]$$

We consider the case $T > 0$ with similar conclusions for the case $T = 0$.

We must show

$$\sup_x \int_{|y-x| \geqslant D} dy \, |(\Omega, [J_k(x), \dot{J}_k(y)] \Omega)| \underset{D \nearrow \infty}{\longrightarrow} 0$$

with

$$J_k(x) = \int d^n y \, h(x-y) j_k(y)$$

Now

$$i[H, j_k(x)] = \frac{\partial S_{kl}(x)}{\partial x_l} - \Psi^\dagger(x) \int d^n y \, \partial_k V(x-y) \rho(y) \Psi(x)$$

where

$$S_{kl}(x) = -\tfrac{1}{2}\left[\partial_k \Psi^\dagger(x) \partial_l \Psi(x) + \partial_l \Psi^\dagger(x) \partial_k \Psi(x) \right] + \frac{\delta_{kl}}{4} \left[\nabla^2 \Psi^\dagger(x) \Psi(x) + 2 \underline{\nabla} \Psi^\dagger(x) \cdot \underline{\nabla} \Psi(x) + \Psi^\dagger(x) \nabla^2 \Psi(x) \right]$$

where

$$\partial_k V(\xi) = \frac{\partial V(\xi)}{\partial \xi_k}$$

Now since S_{kl} is a local term it will not contribute to the estimate, as $D \nearrow \infty$

$$[j_k(x), [V, j_k(y)]] = 2\Big[- V_{kk}(x-y) : \rho(x)\rho(y): + \int d^n z \, V_{kk}(z-y) : \rho(z)\rho(y) : \delta(x-y) \Big]$$

where

$$V_{kk}(\xi) = \frac{d^2V(\xi)}{d\xi_k^2}$$

Thus the only term of importance in the estimate is

$$\sup_x \int_{|y-x| \geqslant D} dy \cdot |V_{kk}(x-y)| \, |(\Omega, \, :\rho(x)\rho(y): \Omega)|$$

Thus if

$$\sup_{x,y} |(\Omega, \, :\rho(x)\rho(y): \Omega)| < \infty$$

and

$$\int dx \, |V_{kk}(x)| < \infty$$

the result follows.

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